Outline

1 Background

2 Young-Laplace Eqn

3 Deriving a Boundary Condition

4 Computing the solutions and eigenfrequencies

5 Closing Remarks
My Goal

- To model the waves which form on the surface of a water balloon impinging on a surface
  - Look at acoustic (pressure) waves created within the water balloon
  - Look at waves formed from deformation of the balloon surface

Figure: Waves formed on a water balloon surface
Previous approach looked at an acoustic driving force driving oscillations on a membrane.

This is mathematically complicated: two coupled PDEs (the acoustic pressure wave, and the wave equation on the surface).

Better approach: try modelling the surface force as the surface tension of a non-wetting droplet.

This is governed by the Young-Laplace Equation.

**Figure**: A travelling Gaussian isobar impinging from below a membrane.
Brief Review

Fluid mechanics: describe the velocity of “elements” of the fluid, $\vec{u}$

If irrotational flow: $\nabla \times \vec{u} = 0$, therefore $\vec{u} = \nabla \psi$

$\psi$ is called the velocity potential and it satisfies Laplace’s Equation

$$\nabla^2 \psi = 0$$

Goal: Solve the Laplace equation for the a droplet.

- Velocity potential of fluid at surface of balloon will give velocity of balloon surface
- Need a boundary condition to solve the Laplace Equation
Young-Laplace Equation

The Young-Laplace Equation describes the pressure difference at the surface between two fluid media:

$$\Delta p = \gamma \Omega$$

- $\Delta p = p_1 - p_2$ where $p_1$ is pressure in medium 1 and $p_2$ is pressure in medium 2
- $\gamma$ is the surface tension (units J/m$^2$ or N/m)
- $\Omega$ is the curvature $(1/R_1 + 1/R_2)$ where $R_1$ and $R_2$ are the radii of curvature of the surface in two orthogonal directions

**Figure**: A fluid-fluid interface between water and air ($\gamma \approx 72$ mN/m)
A Slightly Deformed Sphere

Need to calculate the curvature of a sphere that is slightly deformed

Consider radius of slightly deformed sphere to be

\[ r(\theta, \phi) = R + \zeta(\theta, \phi) \]

- \( R \) is the original radius
- \( \zeta \) is a small deviation from \( R \)

Figure: Near-sphere, with slight changes in radius \( \zeta \)
What is \( \frac{1}{R_1} + \frac{1}{R_2} \)?

Can be calculated by equating the infinitesimal change in the surface area

\[
\delta A = \int \int \delta \zeta \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dA
\]

\( \delta \zeta \) – small change in radius.

Alternatively, calculating the surface area of the deformed sphere:

\[
A = \int \int (R + \zeta) \sqrt{1 + \nabla^2 r \delta \zeta} dA
\]

which for small change \( \delta \zeta \) becomes

\[
\delta A = \int \int \left[ \frac{2}{R} - \frac{2\zeta}{R^2} - \frac{1}{R^2} \left( \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) \right) \right] \delta \zeta dA
\]

equating the integrands we get...
Surface Pressure and Fluid Pressure

Young-Laplace Equation becomes

\[ \Delta p = p_f - p_{air} = \gamma \left[ \frac{2}{R} - \frac{2\zeta}{R^2} - \frac{1}{R^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \phi^2} \right) \right] \]

- \( p_{air} \) is constant, ambient
- \( p_f = -\rho \frac{\partial \psi}{\partial t} \)

At the surface \( \frac{\partial \zeta}{\partial t} = \frac{\partial \psi}{\partial r} \). Differentiate the above w.r.t. time and substitute:

The boundary condition

\[ \rho \frac{\partial^2 \psi}{\partial t^2} - \frac{\gamma}{R^2} \left[ 2 \frac{\partial \psi}{\partial r} + \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) \right] = 0 \]
The pressure on the surface isn’t $p_{air}$ at every point of the sphere. At the bottom there is a Dirac delta pressure

$$P_f = \delta(r = R, \theta = \pi, \phi = 0)$$

this changes the boundary condition equation (adds an extra term)

**Figure**: A sphere droplet resting on a plane
Solution of Laplace’s Equation

Look for a solution

\[ \psi = \exp(-i\omega t) f(r, \theta, \phi) \]

so

\[ \nabla^2 \psi = 0 \]

\[ \nabla^2(\exp(-i\omega t)f(r, \theta, \phi)) = \exp(-i\omega t) \nabla^2 f(r, \theta, \phi) = \nabla^2 f(r, \theta, \phi) = 0 \]

so \( f \) must solve Laplace’s Equation.
Well known solution to Laplace’s Equation in spherical coordinates:

\[ f(r, \theta, \phi) = r^l Y_{l,m}(\theta, \phi) \]

Also, \( Y_{l,m} \) are eigenfunctions of the Laplacian:

\[ \nabla^2 Y_{l,m} = -l(l + 1) Y_{l,m} \]
Plugging in our solution

The boundary condition

\[ \rho \frac{\partial^2 \psi}{\partial t^2} - \frac{\gamma}{R^2} \left[ 2 \frac{\partial \psi}{\partial r} + \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) \right] = 0 \]

with

\[ \psi = \exp(-i\omega t) r^l Y_{l,m}(\theta, \phi) \]

reduces to

\[ \omega_i^2 = \frac{\gamma(l-1)(l+2)}{\rho R^3} \]

or, when the expansion of the contact force is included

\[ \omega_i^2 = \frac{\gamma(l-1)(l+2)}{\rho R^3 \left(1 + \sqrt{(2l + 1)/4\pi}\right)} \]
Summary

- Surface effects should be treated as surface tensions, to avoid two coupled PDEs
- Young-Laplace equation governs pressure differences caused by surface tension
- The Y-L equation can be used to get a boundary condition of the Laplace equation for fluid velocity potential
There are some problems with this model

- Applied pressure is not just at a point, but grows with time
- Difficult to determine “surface tension” of a balloon – wouldn’t expect this to be equal to the elastic tension
- This is theory is for *small* droplets for which gravity is negligible to capillary action

However, this my best attempt yet

- Neatly ties together the surface term and the internal velocity field
- Reduces to the easily solved Laplace equation, for the velocity potential
Future Work

- Account for gravity waves in the water balloon
- Treat contact force as an expanding area as a function of time, rather than point
- Compare measured values to predicted
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References:

- Rayleigh, on the Capillary Phenomena of Jets, Proceedings of the Royal Society, (1879)